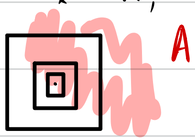
Descriptive Set Theory
Lecture 16

A strengthening of the 95% lemma is the following theorem, for which we need some terminology. For a meas. set $A \subseteq \mathbb{R}^d$, and $x \in \mathbb{R}$, let we define the density of A at x by:



$$d_A(x) := \lim_{\varepsilon \to 0} \frac{\lambda(I_\varepsilon(x) \cap A)}{\lambda(I_\varepsilon(x))},$$

where $I_\varepsilon(x)$ is the open cube centered at x of side length ε, i.e. $I_\varepsilon(x) = \prod_{i=1}^{d} (x_i - \frac{\varepsilon}{2}, x_i + \frac{\varepsilon}{2})$. We don't know yet whether this limit exists or not, so d_A is a partial function $\mathbb{R}^d \to [0,1]$. The set of density 1 points of A is denoted by $\mathcal{D}(A)$ i.e.

$$\mathcal{D}(A) := \{x \in \mathbb{R}^d : d_A(x) = 1\}.$$

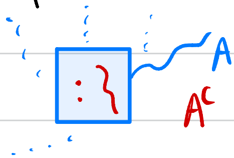
By definition, if $A =_\lambda A'$, then $d_A = d_{A'}$, in particular, $\mathcal{D}(A) = \mathcal{D}(A')$.

Lebesgue differentiation. For any meas. $A \subseteq \mathbb{R}^d$, $\mathbb{1}_A = d_A$ a.e. In particular, $A =_\lambda \mathcal{D}(A)$.

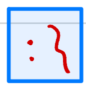
Just like $\mathcal{U}(A)$ in the Baire measurable context, $\mathcal{D}(A)$ is a canonical representative for the $=_\lambda$-class of A.

We now define a new – finer – topology on $\mathbb{R}^d$ that turns Lebesgue measurability into Baire measurability.

Def. The *Lebesgue density topology* is the one in which the open sets are the measurable sets $A \subseteq \mathbb{R}^d$ s.t. $A \subseteq \mathcal{D}(A)$.



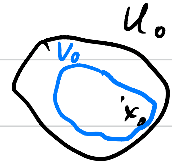
then



is Lebesgue open.

Note that it is challenging to prove that this is indeed a topology because one has to show that an arbitrary union of Lebesgue open sets is still measurable. This is done via a Vitali covering argument.

Note that this topology contains the standard topology on $\mathbb{R}^d$, indeed, for any open cube C, $\mathcal{D}(C) = C$. In fact, it is strictly finer: take any open set and remove any null set, the resulting set is Lebesgue open but typically, it won't be Euclid open. One sees that it's not separable because the complement of any ctbl set is Lebesgue open. Although this top is not separable and not even metrizable, it is strong Choquet, i.e. in the following game, P2 has a winning strategy:



	strong		
P1	(U_0, x_0)	(U_1, x_1)	...
P2	V_0	V_1	

where $U_{n+1} \subseteq V_n \subseteq U_n$ and $V_n \ni x_n$. P2 wins $\iff \bigcap_n V_n \neq \emptyset$.

The main point is that this game enables running the proof of Baire category theorem, yielding that strong Choquet, in fact just Choquet, spaces are Baire. Strong Choquet condition + the fact that Lebesgue density topology contains a Polish topology, makes the proof of the Banach–Mazur theorem go through, characterizing Baire measurability for this topology.

Obs. For any $A \subseteq \mathbb{R}^d$, TFAE.

(1) A is Lebesgue nowhere dense.

(2) A is λ-null.

(3) A is meager.

Proof. (3) $\iff$ (2). Follows from (1) $\iff$ (2).

(1) $\implies$ (2). If A is nowhere dense, then its closure $\bar{A}^L$

has empty interior. But $\bar{A}^L$ is measurable so $\mathrm{int}_L(\bar{A}^L) = \bar{A}^L \cap D(\bar{A}^L) = \emptyset$, but $\bar{A}^L \cap D(\bar{A}^L) =_\lambda \bar{A}^L$, so $\bar{A}^L$ is null, hence A is null.

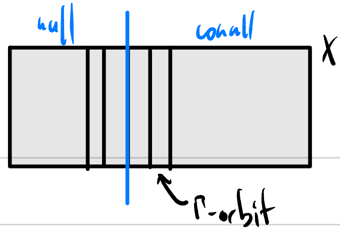
(2) $\Rightarrow$ (1). For any Lebesgue open set U, $U \setminus A$ is still Lebesgue open and nonnull if U is nonnull, so A is Lebesgue nowhere dense. $\square$

Remark. The only null Lebesgue open set is $\emptyset$.

Cor. Lebesgue mearable sets are precisely the Baire measurable sets in the Lebesgue density topology. Thus, the Banach–Mazur game will characterize Lebesgue measurability.

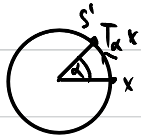
Applications to group actions. Let Γ be a ctbl gp acting on a Polish space X in a Borel way, i.e. if $B \subseteq X$ is Borel then $\gamma \cdot B$ is also Borel $\forall \gamma \in \Gamma$. For a Borel measure μ on X, call this action μ-ergodic if every Γ-invariant Borel set is either null or conull. Here, a set $A \subseteq X$ is Γ-invariant if $\Gamma \cdot A = A$, equivalently,

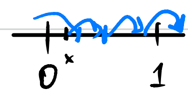
A is a union of orbits.



Examples of actions. (a) $\mathbb{Q} \curvearrowright \mathbb{R}$ with Lebesgue measure.

(b) Rotation of S^1 by angle α, i.e. $\mathbb{Z} \curvearrowright S^1$ by letting 1 act as the transformation $T_\alpha : S^1 \to S^1$, where $T_\alpha x$ is $x \cdot e^{2\pi\alpha i}$. We think of S^1 as $[0,1)$, in which case this T_α becomes $x + \frac{\alpha}{2\pi} \mod 1$.





(c) Let (Y, ν) be any Polish space equipped with a Borel prob measure ν, e.g. $(2, \nu)$, where $\nu = p\delta_1 + (1-p)\delta_0$.

Then for any ctbl group Γ, it acts on $X := Y^\Gamma$ by shift: for $\gamma \in \Gamma$, $x \in X$,

$$\gamma \cdot x := (x_{\eta\gamma})_{\eta \in \Gamma}$$

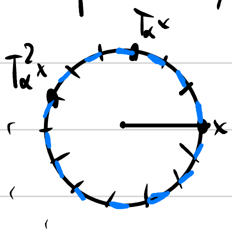
One could also define another shift action by using left multiplication by γ^{-1}, i.e.

$$\gamma \cdot x := (x_{\gamma^{-1}\eta})_{\eta \in \Gamma}.$$

When X is equipped with the power measure $\mu := \nu^\Gamma$, we call this action a Bernoulli shift or a Bernoulli action.

Obs. Rational rotation, i.e. when $\frac{\alpha}{2\pi} \in \mathbb{Q}$, is not ergodic. In fact, it is periodic, i.e. every orbit is finite.

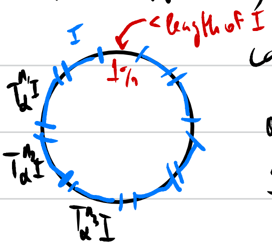
Proof.



Let $\alpha := \frac{n}{m} 2\pi$, then $T_\alpha^{\operatorname{lcm}(n,m)}(x) = x \ \forall x \in S^1$. A is invariant under T_α and $\mu(A) = \frac{1}{2}$. $\square$

Prop. Irrational rotation is ergodic.

Proof. Let $\alpha/2\pi \notin \mathbb{Q}$. It's an exercise in Euclidean algorithm to show that every orbit is dense in S^1. Now let $A \subseteq S^1$ be an invariant Borel nonnull null set and we show that A is conull. We'll show that A is 97% of S^1. By the 99% lemma (or rather its proof) $\exists$ interval I of length $< 1\%$ of S^1 that is 99% A. By density of each orbit, we can cover 98% of S^1 by disjoint translates of I. But each translate $T_\alpha^{n_i} I$ is still 99% A because A is invariant. Thus,



S^1 is 97% A. $\square$

Prop. Irrational rotation is generically ergodic.

Proof. It's a continuous action and has a dense orbit (in fact all of them). Thus by homework, using the 100% lemma, it is generically ergodic. $\square$